A Study of M-Projective Curvature Tensor $\overline{W}^i_{jkh}$ in $G\mathcal{B}K-5RF_n$ Via Lie Derivative

Adel M. Al-Qashbari [1], Alaa A. Abdallah [*2] & Saeedah M. Baleedi [3]
[1]Department. of Mathematices, Education Faculty, University of Aden, Yemen
Department of Engineering, Faculty of Engineering and Computing,
University of Science & Technology-Aden, Yemen.
Email: a.alqashbari@ust.edu
[2,3]Department of Mathematics, Education Faculty, University of Abyan, Zingibar, Yemen
Email: ala733.ala00@gmail.com : saeedahbaleedi@gmail.com
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Abstract
In this paper, we investigate the M-projective curvature tensor $\overline{W}^i_{jkh}$ in generalized fifth recurrence Finsler space for Cartan's fourth curvature tensor in sense of Berwald by using Lie-derivative. Our study aims to provide new insights into the geometric properties of these spaces and the behavior of the M-projective curvature tensor $\overline{W}^i_{jkh}$ under Lie differentiation. By employing the Lie derivative, we are able to derive new identities and relations involving the M-projective curvature tensor $\overline{W}^i_{jkh}$. We get relationships between some tensors when the M-projective curvature tensor $\overline{W}^i_{jkh}$ is collineation along a vector field $v^i(x)$. Furthermore, we obtain the Berwald covariant derivative of fifth order for some tensors are vanishing and we establish various identities on Lie-derivative in $G\mathcal{B}K-5RF_n$.

Keywords. *M-projective curvature tensor $\overline{W}^i_{jkh}$, Generalized $\mathcal{B}K$ –fifth recurrent space, Berwald covariant derivative of fifth order, Lie-derivative.*

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1. Introduction and Preliminaries

The study of curvature tensors plays a fundamental role in differential geometry. Among the various curvature tensors, the projective curvature tensor has been extensively investigated due to its connections to projective geometry and its applications in physics. In recent years, there has been a growing interest in generalizing the notion of projective curvature tensor to obtain new geometric invariants and explore their properties.

In this context, the M-projective curvature tensor emerged as a significant generalization of the classical projective curvature tensor. This tensor studied in various space-time models and has possessed interesting properties.

The Lie-derivative in generalized fifth recurrence Finsler space for Carton's fourth curvature tensor in sense of Berwald introduced by AL-Qashbari and Baleedi [10]. Ali et al. [6] studied a Lie - derivative of M- projective curvature tensor and established some properties of this curvature tensor. Further, Gouin [19] introduced some remarks on the Lie-derivative. Opondo [22] studied the Lie-derivative on W-curvature tensor in recurrent and bi-recurrent

Finsler space. The Lie-derivative of forms and its application was investigated by authors [20, 23].

The generalized recurrence, birecurrence and trirecurrence properties for various curvature tensors in sense of Berwald have been discussed by [2, 3, 4, 5, 11, 12, 13, 17, 21]. The generalized recurrent Finsler spaces of higher orders have been studied by [8, 18]. Let us explore the infinitesimal transformation point given by [10]

(1.1) $x^{-i} = x^i + v^i(x)\varepsilon$.

Where ε is an infinitesimal point constant and $v^i(x)$ is acontravariant vector field independent of directional arguments and dependent on positional coordinates x^i only, also $v^i(x) \neq 0$. Infinitesimal method is a tool that leads to Lie-derivatives. The symbol L_v denote the Lie-differentiation operator with respect to the transformation (1.1). The Lie - derivative of a vector field x^i in sense of Berwald is given by

(1.2) $L_v x^i = v^j \mathcal{B}_j x^i - x^j \mathcal{B}_j v^i + (\dot{\partial}_j x^i)\mathcal{B}_s v^j y^s$.

The Lie - derivative of a general mixed tensor field $T^i_j(x, \dot{x})$ expressed in the form

(1.3) $$L_v T^i_{jkh} = v^m \mathcal{B}_m T^i_{jkh} - T^m_{jkh}\mathcal{B}_m v^i + T^i_{mkh}\mathcal{B}_j v^m + T^i_{jmh}\mathcal{B}_k v^m + T^i_{jkm}\mathcal{B}_h v^m + \dot{\partial}_m T^i_{jkh}\mathcal{B}_r v^m y^r .$$

The vector y^i is Lie-invariant i.e.

(1.4) $L_v\, y^i = 0$.

The metric tensor g_{ij} and the Kronecker delta δ^i_h are given by [1, 16]

(1.5) a) $g_{ij}\, y^j = y_i$, b) $\delta^i_i = n$, c) $g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & if\ \ j = k\,, \\ 0 & if\ \ j \neq k. \end{cases}$

d) $\delta^i_k\, y_i = y_k$, e) $\delta^j_h = \dot{\partial}_h y^j$, f) $g_{hk} = \dot{\partial}_h y_k$ and j) $\dot{\partial}_j y^j = 1$.

The metric tensor, Cartan's connection parameters and Berwald's connection parameters are symmetric in their lower indices and they are positive homogeneous of degree zero in the directional arguments.

Berwald's covariant derivative of the vectors y^i and y_i vanish identically, i.e. [15]

(1.6) a) $\mathcal{B}_k\, y^i = 0$ and b) $\mathcal{B}_k\, y_i = 0$

The Cartan's fourth curvature tensor K^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} are skew-symmetric it their last two lower indices and satisfy the following relations

(1.7) a) $H^i_{kh} = K^i_{jkh}\, y^j = R^i_{jkh}\, y^j$, b) $K^i_{jkh} = R^i_{jkh} - C^i_{js} H^s_{kh}$ and c) $R^i_{jkh} g^{jk} = R^i_h$.

The torsion tensor H^i_{kh} , deviation tensor H^i_h and torsion tensor C^i_{jk} satisfy the following relations

(1.8) a) $H^i_{kh} y^k = H^i_h$, b) $H^i_{jk} y_i = 0$ and c) $C^i_{jk}\, y^j = 0$.

A Finsler space whose Berwald connection parameter G^i_{kh} is independent of y^i is called an affinely connected space (Berwald space). Thus, an affinely connected space is characterized by one of the equivalent conditions [14]

(1.9) a) $\mathcal{B}_k g_{ij} = 0$ and b) $\mathcal{B}_k\, g^{ij} = 0$.

Let us explore a generalized $\mathcal{B}K$ –fifth recurrent Finsler space satisfying the relations [9]

(1.10) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm} K^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k\, g_{jh}\right) - c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl}\right) - e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}\right) - 2b_{qlnm} y^r \mathcal{B}_r\left(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs}\right) .$$

(1.11) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm} H^i_{kh} + b_{sqlnm}\left(\delta^i_h y_k - \delta^i_k y_h\right)$.

(1.12) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = a_{sqlnm} H^i_h + b_{sqlnm}\left(\delta^i_h F^2 - y^i y_h\right)$.

(1.13) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{sqlnm}R^i_{jkh} + b_{sqlnm}(\delta^i_h g_{jk} - \delta^i_k\, g_{jh})$
$- c_{sqlnm}(\delta^i_h C_{jkn} - \delta^i_k\, C_{jhn}) - d_{sqlnm}(\delta^i_h C_{jkl} - \delta^i_k\, C_{jhl})$
$- e_{sqlnm}(\delta^i_h C_{jkq} - \delta^i_k\, C_{jhq}) - 2b_{qlnm}y^r\mathcal{B}_r(\delta^i_h C_{jks} - \delta^i_k\, C_{jhs})$.

If and only if

(1.14) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^i_{jt}H^t_{kh}) - a_{sqlnm}(C^i_{jt}H^t_{kh}) = 0.$

Taking the Lie-derivative of both sides of [(1.5)d] and using [(1.5)c] when $i \neq j$, we get

(1.15) $L_v y_k = 0$.

Taking the Lie-derivative of both sides of [(1.5)a] and using (1.15) and (1.4), we get

(1.16) $L_v g_{ij} = 0$.

M-projective curvature tensor collineation along a vector field $v^i(x)$ satisfied the relation

(1.17) $L_v \overline{W}^i_{jkh} = 0$.

The M-projective curvature tensor $\overline{W}^i_{jkh}$ is given by [7]

(1.18) $\overline{W}^i_{jkh} = R^i_{jkh} - \frac{1}{6}(R_{jk}\delta^i_h - R_{jh}\delta^i_k + g_{jk}R^i_h - g_{jh}R^i_k)$.

By applying (1.3) on the Cartan's fourth curvature tensor K^i_{jkh} and $h(v)$ −torsion tensor H^i_{kh}, using [(1.6)a] and [(1.5)e,c] when $r \neq m$, we get

(1.19) $L_v K^i_{jkh} = v^m\mathcal{B}_m K^i_{jkh} - K^m_{jkh}\mathcal{B}_m v^i + K^i_{mkh}\mathcal{B}_j v^m + K^i_{jmh}\mathcal{B}_k v^m + K^i_{jkm}\mathcal{B}_h v^m$,

and

(1.20) $L_v H^i_{kh} = v^m\mathcal{B}_m H^i_{kh} - H^m_{kh}\mathcal{B}_m v^i + H^i_{mh}\mathcal{B}_k v^m + H^i_{km}\mathcal{B}_h v^m$, respectively.

Taking the Lie-derivative of both sides of [(1.7) a], we get

(1.21) $L_v H^i_{kh} = L_v\, K^i_{jkh}\, y^j$.

Using (1.4) in above equation, we get

(1.22) $L_v\, H^i_{kh} = y^j L_v\, K^i_{jkh}$.

Using (1.19), [(1.7) a] and (1.4) in above equation, we get

(1.23) $L_v H^i_{kh} = v^m\mathcal{B}_m H^i_{kh} - H^m_{kh}\mathcal{B}_m v^i + K^i_{mkh}\mathcal{B}_j v^m y^j + H^i_{mh}\mathcal{B}_k v^m + H^i_{km}\mathcal{B}_h v^m$.

In view of (1.23) and (1.20), we get

(1.24) $K^i_{mkh}\mathcal{B}_j v^m y^j = 0$.

Using [(1.6)a] in above equation and since $K^i_{mkh} \neq 0$ and $y^j \neq 0$, we get

(1.25) $\mathcal{B}_j v^m = 0$.

The main objective of this paper is studing of the M-projective curvature tensor in $G\mathcal{B}K - 5RF_n$. By utilizing the Lie-derivative, we are able to derive new identities and relations that provide deeper insights into the geometric properties of $G\mathcal{B}K - 5RF_n$.

2. A Lie-derivative of M-projective curvature tensor in $G\mathcal{B}K - 5RF_n$

Let us explore a generalized $\mathcal{B}K$ −fifth recurrent space that Cartan's fourth curvature tensor K^i_{jkh} is defined as (1.10). Taking the Lie - derivative of both sides of (1.18) and using [(1.5)c], we get

(2.1) $L_v \overline{W}^i_{jkh} = L_v R^i_{jkh} - \frac{1}{6} L_v(g_{jk}R^i_h - g_{jh}R^i_k)$.

By applying (1.3) on the M-projective curvature tensor $\overline{W}^i_{jkh}$ and Cartan's third curvature tensor R^i_{jkh}, using the result equations and (1.25) in (2.1), we get

(2.2) $v^m\mathcal{B}_m \overline{W}^i_{jkh} = v^m\mathcal{B}_m R^i_{jkh}$,

if the tensor $g_{jk}R^i_h$ is symmetric in its two lower indices *h* and *k*.

Taking Berwald covariant derivative of fourth order for (2.2) with respect to x^n, x^l, x^q and x^s, using (1.25) and since $v^m \neq 0$, we get

(2.3) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\bar{W}^i_{jkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}$,

if

(2.4) $g_{jk}R^i_h = g_{jh}R^i_k$.

Thus, we conclude

Theorem 2.1. *In $G\mathcal{B}K - 5RF_n$, the Berwald covariant derivative of fifth order for M-projective curvature tensor $\bar{W}^i_{jkh}$ and Cartan's third curvature tensor R^i_{jkh} are equal if the tensor $g_{jk}R^i_h$ is symmetric in its two lower indices h and k.*

Using [(1.5)c] in (1.13) and using the result equation in (2.3), we get

(2.5) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\bar{W}^i_{jkh} = a_{sqlnm}R^i_{jkh}$,

Let as assume $\bar{W}^i_{jkh} = R^i_{jkh}$, then above equation can be written as

(2.6) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\bar{W}^i_{jkh} = a_{sqlnm}\bar{W}^i_{jkh}$,

Thus, we conclude

Theorem 2.2. *In $G\mathcal{B}K - 5RF_n$, the M-projective curvature tensor $\bar{W}^i_{jkh}$ behaves as fifth recurrent if it equals Cartan's third curvature tensor R^i_{jkh} [provided (1.14) and (2.4) hold].*

Using [(1.7)b] in (2.1), we get

(2.7) $L_v\bar{W}^i_{jkh} = L_vK^i_{jkh} + L_vC^i_{js}H^s_{kh} - \frac{1}{6}L_v\left(g_{jk}R^i_h - g_{jh}R^i_k\right)$.

By applying (1.3) on the M-projective curvature tensor $\bar{W}^i_{jkh}$, Cartan's fourth curvature tensor K^i_{jkh} and tensor $C^i_{js}H^s_{kh}$, then using the result equation (1.25) in (2.7), we get

(2.8) $v^m\mathcal{B}_m\bar{W}^i_{jkh} = v^m\mathcal{B}_mK^i_{jkh}$,

if and only if

(2.9) $\mathcal{B}_m(C^i_{js}H^s_{kh}) = 0$.

Taking Berwald covariant derivative of fourth order for (2.8) with respect to x^n ,x^l, x^q and x^s, using (1.25), we get

(2.10) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\bar{W}^i_{jkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_mK^i_{jkh}$.

Thus, we conclude

Theorem 2.3. *In $G\mathcal{B}K - 5RF_n$, the Berwald covariant derivative of fifth order for M-projective curvature tensor $\bar{W}^i_{jkh}$ and Cartan's fourth curvature tensor K^i_{jkh} are equal [provided (2.4) and (2.9)hold].*

Using [(1.5)c] in (1.10) and using the result equation in (2.10), we get

(2.11) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\bar{W}^i_{jkh} = a_{sqlnm}K^i_{jkh}$,

Let as assume $\bar{W}^i_{jkh} = K^i_{jkh}$, then above equation can be written as

(2.12) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\bar{W}^i_{jkh} = a_{sqlnm}\bar{W}^i_{jkh}$,

Thus, we conclude

Corollary 2.1. *In $G\mathcal{B}K - 5RF_n$, the M-projective curvature tensor $\bar{W}^i_{jkh}$ behaves as fifth recurrent if it equals Cartan's fourth curvature tensor K^i_{jkh} [provided (2.4) and(2.9) hold].*

Using (2.4) in (2.7), we get

(2.13) $L_v \bar{W}^i_{jkh} = L_v K^i_{jkh} + L_v C^i_{js} H^s_{kh}$.

Using (1.3) and (1.25) in above equation , we get

(2.14) $\mathcal{B}_m \bar{W}^i_{jkh} = \mathcal{B}_m K^i_{jkh} + \mathcal{B}_m C^i_{js} H^s_{kh}$.

Multiplying above equation by y^j , using [(1.6)a], [(1.7)a] and [(1.8)c], we get

(2.15) $\mathcal{B}_m (\bar{W}^i_{jkh} y^j) = \mathcal{B}_m H^i_{kh}$.

Taking Berwald covariant derivative of fourth order for (2.15) with respect to x^n, x^l, x^q and x^s, we get

(2.16) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m \left(\bar{W}^i_{jkh} y^j\right) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H^i_{kh}$.

Using (1.11) and [(1.5)c] in above equation, we get

(2.17) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m \left(\bar{W}^i_{jkh} y^j\right) = a_{sqlnm} H^i_{kh}$,

Let as assume $\bar{W}^i_{jkh} y^j = H^i_{kh}$, then above equation can be written as

(2.18) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m (\bar{W}^i_{jkh} y^j) = a_{sqlnm} (\bar{W}^i_{jkh} y^j)$,

Thus, we conclude

Theorem 2.4. *In $G\mathcal{B}K - 5RF_n$, the tensor ($\bar{W}^i_{jkh} y^j$) behaves as fifth recurrent if it equals the h(v)-torsion tensor H^i_{kh}.*

Multiplying (2.16) by y^k , using [(1.6)a] and [(1.8)a], we get

(2.19) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m \left(\bar{W}^i_{jkh} y^j y^k\right) = \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m H^i_h$.

Using (1.12) in above equation, we get

(2.20) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m \left(\bar{W}^i_{jkh} y^j y^k\right) = a_{sqlnm} H^i_h$,

if and only if

(2.21) $b_{sqlnm}\left(\delta^i_h F^2 - y^i y_h\right) = 0.$

Let as assume $\bar{W}^i_{jkh} y^j y^k = H^i_h$, then equation (2.20) can be written as

(2.22) $\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m \left(\bar{W}^i_{jkh} y^j y^k\right) = a_{sqlnm} \left(\bar{W}^i_{jkh} y^j y^k\right)$,

Thus, we conclude

Theorem 2.5. *In $G\mathcal{B}K - 5RF_n$, the tensor ($\bar{W}^i_{jkh} y^j y^k$) behaves as fifth recurrent if it equals the deviation tensor H^i_h [provided (2.21)holds].*

Let the tensor $g_{jh} R^i_k$ is skew-symmetric in its two lower indices *h* and *k*, then the equation (2.1) can be written as

(2.23) $L_v \bar{W}^i_{jkh} = L_v R^i_{jkh} - \frac{1}{3} L_v (g_{jk} R^i_h)$.

Using (1.3), (1.25) and (1.16) in above equation , we get

(2.24) $\mathcal{B}_m \bar{W}^i_{jkh} = \mathcal{B}_m R^i_{jkh} - \frac{1}{3}\, g_{jk} \mathcal{B}_m R^i_h$.

Using [(1.7)c] and [(1.9)b] in above equation, we get

(2.25) $\mathcal{B}_m \bar{W}^i_{jkh} = \mathcal{B}_m R^i_{jkh} - \frac{1}{3}\, g_{jk}\, g^{jk} \mathcal{B}_m R^i_{jkh}$.

Using [(1.5)c,b] in above equation, we get

(2.26) $\mathcal{B}_m \bar{W}^i_{jkh} = (1 - \frac{n}{3})\, \mathcal{B}_m R^i_{jkh}$.

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s, we get

(2.27) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\bar{W}^i_{jkh} = (1-\frac{n}{3})\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}$.

Which can be written as

(2.28) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(\bar{W}^i_{jkh} - (1-\frac{n}{3})R^i_{jkh}\right) = 0$.

Thus, we conclude

Theorem 2.6. *In $G\mathcal{B}K-5RF_n$, the Berwald covariant derivative of fifth order for the tensor $\left(\bar{W}^i_{jkh} - (1-\frac{n}{3})R^i_{jkh}\right)$ is vanishing if the tensor $g_{jh}R^i_k$ is skew-symmetric in its two lower indices h and k.*

Now, using (1.17) in (2.1), we get

(2.29) $L_v R^i_{jkh} - \frac{1}{6}L_v\left(g_{jk}R^i_h - g_{jh}R^i_k\right) = 0$.

Multiplying above equation by y^j, using (1.4), [(1.5)a] and [(1.7)a], we get

(2.30) $L_v H^i_{kh} = \frac{1}{6}L_v\left(y_k R^i_h - y_h R^i_k\right)$.

Using (1.15), (1.3) and (1.25) in above equation, we get

(2.31) $\mathcal{B}_m H^i_{kh} = \frac{1}{6}\left(y_k\mathcal{B}_m R^i_h - y_h\mathcal{B}_m R^i_k\right)$.

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s, and using [(1.6)b], we get

(2.32) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{1}{6}\left(y_k R^i_h - y_h R^i_k\right)\right]$.

Thus, we conclude

Theorem 2.7. *In $G\mathcal{B}K-5RF_n$, the Berwald covariant derivative of fifth order for $h(v)$ −torsion tensor H^i_{kh} and the tensor $[\frac{1}{6}\left(y_k R^i_h - y_h R^i_k\right)]$ are equal if the M-projective curvature tensor collineation along a vector field $v^i(x)$.*

Using (1.3), (1.25), (1.16) and [(1.5)f] in (2.29), we get

(2.33) $\mathcal{B}_m R^i_{jkh} - \frac{1}{6}\left(\dot{\partial}_j y_k\mathcal{B}_m R^i_h - \dot{\partial}_j y_h\mathcal{B}_m R^i_k\right) = 0$.

Multiplying above equation by y^j, using [(1.5)j], [(1.6)a] and [(1.7)a], we get

(2.34) $\mathcal{B}_m H^i_{kh} - \frac{1}{6}\left(y_k\mathcal{B}_m R^i_h - y_h\mathcal{B}_m R^i_k\right) = 0$.

Multiplying above equation by y_i, using [(1.8)b], we get

(2.35) $y_i[\frac{1}{6}\left(y_k\mathcal{B}_m R^i_h - y_h\mathcal{B}_m R^i_k\right)] = 0$.

Which can be written as

(2.36) $y_k\mathcal{B}_m R^i_h - y_h\mathcal{B}_m R^i_k = 0$.

Taking Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s, using [(1.6)b], we get

(2.37) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(y_k R^i_h) = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(y_h R^i_k\right)$.

Thus, we conclude

Theorem 2.8. *In $G\mathcal{B}K-5RF_n$, the Berwald covariant derivative of fifth order for the the tensor $(y_k R^i_h)$ is symmetric in its two lower indices h and k if the M-projective curvature tensor collineation along a vector field $v^i(x)$.*

In view of [Theorem 2.7] and [Theorem 2.8], we get

(2.38) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = 0$.

Thus, we conclude

Corollary 2.2. *In $G\mathcal{B}K-5RF_n$, the Berwald covariant derivative of fifth order for $h(v)$ −torsion tensor H^i_{kh} is vanishing if the M-projective curvature tensor $\bar{W}^i_{jkh}$ collineation along a vector field $v^i(x)$[provided (2.37)holds].*

In view of [Theorem 2.1] and [Theorem 2.3], we get

(2.39) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} ,$$

Thus, we conclude

Corollary 2.3 *In $G\mathcal{B}K-5RF_n$, the Berwald covariant derivative of fifth order for the Cartan's third curvature tensor R^i_{jkh} and the Cartan's fourth curvature tensor K^i_{jkh} are equal [provided (2.4) and (2.9) hold].*

3. Conclusions

This paper established new identities of the Lie derivative for the M-projective curvature tensor $\bar{W}^i_{jkh}$ in generalized fifth-order recurrent Finsler spaces. The authors established conditions under which the Berwald covariant derivative of fifth order for the M-projective curvature tensor $\bar{W}^i_{jkh}$, Cartan's third curvature tensor R^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} are equal. We proved that the M-projective curvature tensor $\bar{W}^i_{jkh}$, the tensors $\bar{W}^i_{jkh}y^j$ and $\bar{W}^i_{jkh}y^jy^k$ behave as a fifth recurrent under specific conditions. Also, we introduced relationships between some tensors when the M-projective curvature tensor $\bar{W}^i_{jkh}$ is collineation along a vector field $v^i(x)$. The Berwald covariant derivative of fifth order for the subtraction two tensors is vanishing under certain condition has been obtained.

This study advances our understanding of the Lie derivative's role in generalized fifth-order recurrent Finsler geometry, opening avenues for future research.

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